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Parametric Polynomial Curves of Local Approximation of Order 8

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Abstract. Parametric approximation of curves offers the possibility of increasing the order of approximation by using the additional parameters in the parametrization of the curve. This has been studied in several papers, see e.g.[1-8]. The resulting problems are highly nonlinear. Here the cases of approximation order $\mathcal{O}(h^8)$ are studied which need piecewise quartic curves in the plane and piecewise quintic curves in space.

§1. Introduction

A general conjecture concerning the local approximation order by polynomial curves can be formulated as follows (see e.g. Mørken-Scherer [6]):

- For sufficiently small $h > 0$ and a sufficiently smooth curve $\underline{f}(t) : t \in [0,h] \to \underline{f}(t) \in \mathbf{R}^d$, there exists a polynomial curve $\underline{p}(t)$ of degree n and a reparametrization φ of $\underline{f}$ on $[a,b]$ such that

$$\sup_{0\leq t\leq h} \|(\underline{f} \circ \varphi)(t) - \underline{p}(t)\| \leq C(\underline{f})h^m ,$$

where $m := n + 1 + [\frac{n-1}{d-1}]$.

The increased order m is explained in [6] by the principle of degree reducing. It comes from the idea of approximating with an interpolating polynomial curve $\underline{p}(t)$ of degree $m-1$ such that

$$\underline{p}(t_i) = (\underline{f} \circ \phi)(t_i), \qquad 1 \leq i \leq m, \tag{1}$$

for points t_i in $[0,h]$ (multiplicities allowed). The additional parameters occuring via the reparametrization ϕ are used to reduce the degree $m-1$ of $\underline{p}(t)$ by requiring

$$[t_1, \ldots, t_{m-i}](\underline{p}) = [t_1, \ldots, t_{m-i}](\underline{f} \circ \phi) = 0, \quad i = 0, \ldots, k-1, \tag{2}$$

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with k as large as possible. Since we have to normalize ϕ such that $\phi(0) = 0, \phi(h) = h$, there are $m-2$ parameters left at our disposal for this goal. Thus $m-1$ can be reduced to $n = m-1-k$, where $k{\cdot}d \le m-2$ or $k \le (n-1)/(d-1)$.

From classical approximation theory it is then clear that a solution of (1),(2) yields a polynomial curve $\underline{p}(t)$ of degree $m-1-k$ satisfying

$$\sup_{0\le t\le h} \|(\underline{f}\circ\phi)(t) - \underline{p}(t)\| \le h^m \sup_{0\le t\le h} \|D^m(\underline{f}\circ\phi)(t)\|. \tag{3}$$

Thus conjecture ($\bullet$) is true if one can guarantee in addition that the parameters $\phi^{(2)}(0),\dots,\phi^{(m)}(0)$ of a solution of (2) remain bounded for $h\to 0$. This question of stability is also discussed in [6]. Note also that in this case equations (1) can be written as

$$\underline{p}\circ\phi^{-1}(s_i) = \underline{f}(s_i), \quad 1\le i\le m,$$

where the nodes s_i are defined by $s_i := \phi(t_i)$.

The most interesting case of the conjecture is when $(n-1)/(d-1)$ is an integer k, i.e. $m = n+1+k = kd+2$. In this case there are kd equations in (2), and the degree $m-1$ is reduced by k. Then ϕ is determined by

$$[t_1,\dots,t_{n+1+i}](\underline{f}\circ\phi) = 0, \quad i = 1,\dots,k, \tag{4}$$

and $\underline{p}(t)$ by the first $n+1$ equations in (1).

So far the conjecture seems to be proved only in the case $n = d$ or $k = 1$ (see [5–6]) and for $k = 2 = d$ (see [1,3]). Here we treat the next most difficult cases $k = 3, d = 2$ and $k = 2, d = 3$, which amount to six equations in (4), and will lead to quartic curves in the plane and quintic curves in space with approximation rates of order 8, respectively.

§2. Reduction to 2×2 Systems

In Morken -Scherer [6] equations (4) were studied in particular for the Taylor case

$$D^{n+i}(\underline{f}\circ\phi) = 0, \quad i = 1,\dots,k.$$

The crucial point is then the formula of Faa di Bruno in the form given by T. Goodman. It reads (cf. [4])

$$D^l(\underline{f}\circ\phi)(0) = \sum_{j=1}^{l} a_{l,j}\underline{f}^{(j)}(0), \qquad \beta_l := \phi^{(l)}(0), \tag{5}$$

where

$$a_{l,j} = \sum_{l_1+\dots l_j = l,\, l\ge 1} \begin{bmatrix} l \\ l_1\cdots l_j \end{bmatrix} \beta_{l_1}\cdots\beta_{l_j}$$

and

$$\begin{bmatrix} l \\ l_1\cdots l_j \end{bmatrix} = \frac{l!}{l_1!\cdots l_j! m_1!\cdots m_r!}.$$

Here the integer r denotes the number of distinct integers among the $l_1, \ldots, l_j$, where $m_1, \ldots, m_r$ are the multiplicities of them. Specific examples are

$$a_{l,1} = \beta_l, \quad a_{l,2} = \sum_{i=1}^{[l/2]} \binom{l}{i} \beta_i \beta_{l-i}, \qquad a_{l,l} = \beta_1^l.$$

At first we consider the planar case $d = 2$ for $k = 3$. Then the equations (4) specialize to

$$D^j(\underline{f} \circ \phi)(0) = 0, \quad j = 5, 6, 7. \tag{6}$$

Under the normalization $\phi(0) = 0$, $\beta_1 := \phi'(0) = 1$, this yields six equations for the unknowns $\beta_2, ..., \beta_7$. In [8] this system has been reduced to a 2×2 system for β_2, β_3. The basic idea was to simplify system (6) by determining a preliminary reparametrization ψ with $\psi(0) = 0, \psi'(0) = 1$ such that

$$D^{2i+1}(\underline{f} \circ \psi)(0) = 0, \quad i = 1, 2, 3.$$

It can easily be shown that this system is uniquely solvable in the unknowns $\gamma_2, ..., \gamma_7$, where $\gamma_j := \psi^{(j)}(0)$ provided

$$\operatorname{span}(\underline{f}'(0), \underline{f}''(0)) = \mathbb{R}^2. \tag{7}$$

Thus one can assume that $D^3\underline{f}(0) = D^5\underline{f}(0) = D^7\underline{f}(0)$ in (6). Denoting $d_{i,j}$ as the cross product of $f^{(j)}$ and $f^{(i)}$ in $\mathbb{R}^2$, i.e.

$$d_{i,j} := \underline{f}^{(i)} \times \underline{f}^{(j)}.$$

Further straightforward computation (cf. [9]) leads to

Lemma 1. *Under the assumption (7) and the normalization $\phi(0) = 0$, $\beta_1 := \phi'(0) = 1$, the vector $\beta_2, ..., \beta_7$ is a solution of the equations (7) iff β_2, β_3 is a solution of the 2×2 system*

$$\begin{aligned} 0 = d_{6,1} &+ 20 d_{4,1}\beta_3 + 10 d_{2,1}\beta_3^2 \\ &+ 60\beta_2 d_{4,2} + 15(d_{4,1} - 2d_{2,1}\beta_3)\beta_2^2, \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 = 15\beta_2^3 d_{4,1} &+ 75\beta_2^2 d_{4,2} + (3d_{6,1} - 10 d_{4,1}^2/d_{2,1})\beta_2 + d_{6,2} \\ &+ 10(2d_{4,2} + \beta_2 d_{4,1})\beta_3 - 10 d_{2,1}\beta_2\beta_3^2. \end{aligned} \tag{9}$$

The elimination of the variable β_3 in this system along the lines of the resultant method yields an equation of degree 9 in β_2. However, it was overlooked in [8] that the constant term in this equation vanishes, so that it is in essence of degree 8 . Thus, existence of a (real) solution of (8)–(9) cannot be derived in this way. We will close this gap in the next section.

In the **space case** $d = 3$ we have to find a reparametrization ϕ such that

$$D^j(\underline{f} \circ \phi)(0) = 0, \quad j = 6, 7. \tag{10}$$

We simplify this again by determining a reparametrization ψ with $\psi(0) = 0$, $\psi'(0) = 1$, $\psi^{(j)}(0) := \gamma_j$ and

$$D^j(\underline{f} \circ \psi)(0) = 0, \quad j = 4, 7.$$

This is possible since by (5), in the three equations forming $D^4(\underline{f} \circ \psi)(0) = 0$ the coefficients of $\underline{f}', \underline{f}''$ and $\underline{f}'''$ are linear in the unknowns $\gamma_2, \gamma_3, \gamma_4$, and in the latter three are linear with respect to the γ_5, γ_6, γ_7. Therefore, these equations are uniquely solvable under the assumption

$$\operatorname{span}(\underline{f}'(0), \underline{f}''(0), \underline{f}'''(0)) = \mathbb{R}^3, \tag{11}$$

and we can consider (10) without loss under the assumption $D^4(\underline{f})(0) = D^7(\underline{f})(0) = 0$. This yields (with $\beta_1 = 1$) the equations

$$\begin{aligned} 0 = \beta_6 \underline{f}' &+ (6\beta_5 + 15\beta_2\beta_4 + 10\beta_3^2)\underline{f}'' \\ &+ (15\beta_4 + 60\beta_2\beta_3 + 15\beta_2^3)\underline{f}''' + 15\beta_2 \underline{f}^{(5)} + \underline{f}^{(6)}, \end{aligned} \tag{12}$$

$$\begin{aligned} 0 = \beta_7 \underline{f}' &+ (7\beta_6 + 21\beta_2\beta_5 + 35\beta_3\beta_4)\underline{f}'' + (105\beta_2^2 + 35\beta_3)\underline{f}^{(5)} \\ &+ (21\beta_5 + 105\beta_2\beta_4 + 70\beta_3^2 + 105\beta_2^2\beta_3)\underline{f}''' + 21\beta_2 \underline{f}^{(6)}. \end{aligned} \tag{13}$$

The next step is to take in (12) the scalar product with cross products $\underline{f}' \times \underline{f}'', \underline{f}' \times \underline{f}'''$ and $\underline{f}'' \times \underline{f}'''$, respectively. We obtain the equivalent equations

$$0 = (15\beta_4 + 60\beta_2\beta_3 + 15\beta_2^3)d_{1,2,3} + 15\beta_2 d_{5,1,2} + d_{6,1,2} \tag{14}$$

$$0 = (6\beta_5 + 15\beta_2\beta_4 + 10\beta_3^2)d_{2,1,3} + 15\beta_2 d_{5,1,3} + d_{6,1,3} \tag{15}$$

$$0 = \beta_6 d_{1,2,3} + 15\beta_2 d_{5,2,3} + d_{6,2,3}, \tag{16}$$

where

$$(\underline{f}^{(i_1)} \times \underline{f}^{(i_2)}, \underline{f}^{(j)}) := \det(\underline{f}^{(i_1)}, \underline{f}^{(i_2)}, \underline{f}^{(j)}) := d_{i_1,i_2,j}.$$

These equations serve for eliminating the variables β_4, β_5 and β_6 since they appear linearly. Before doing this, we transform (13) into three equivalent scalar equations analogously to (12). We obtain the three equations

$$0 = \beta_7 d_{1,2,3} + (105\beta_2^2 + 35\beta_3)d_{5,2,3} + 21\beta_2 d_{6,2,3}, \tag{17}$$

$$0 = (\beta_6 + 3\beta_2\beta_5 + 5\beta_3\beta_4)d_{2,1,3} + (15\beta_2^2 + 5\beta_3)d_{5,1,3} + 3\beta_2 d_{6,1,3}, \tag{18}$$

$$0 = (3\beta_5 + 15\beta_2\beta_4 + 10\beta_3^2 + 15\beta_2^2\beta_3)d_{3,1,2} + (15\beta_2^2 + 5\beta_3)d_{5,1,2} + 3\beta_2 d_{6,1,2}. \tag{19}$$

Equation (17) determines β_7 directly in terms of β_2, β_3. Now we eliminate β_5, β_6 in (18) via (15),(16). This gives at first

$$0 = -(3\beta_2\beta_5 + 5\beta_3\beta_4)d_{2,1,3} + (15\beta_2^2 + 5\beta_3)d_{5,1,3} + 3\beta_2(d_{6,1,3} + 5d_{5,2,3}) + d_{6,2,3}$$

and then

$$\begin{aligned} 0 = &\frac{15}{2}\beta_2^2 d_{1,2,3} + 5\beta_3 d_{5,1,3} + \frac{5}{2}\beta_2(d_{6,1,3} + 6d_{5,2,3}) + d_{6,2,3} \\ &+ (\frac{15}{2}\beta_2^2\beta_4 + 5\beta_3^2\beta_2)d_{1,2,3} - 5\beta_3\beta_4 d_{1,2,3}. \end{aligned}$$

Now we eliminate the variable β_4 by (14). The result is the equation

$$0 = \tilde{q}_0(\beta_2) + \tilde{q}_1(\beta_2)\beta_3 + \tilde{q}_2(\beta_2)\beta_3^2, \tag{20}$$

where

$$\begin{aligned} \tilde{q}_0(\beta_2) := \quad &(15/2)d_{1,2,3}\beta_2^5 - (15/2)d_{1,2,5}\beta_2^3 + [(15/2)d_{1,3,5} - (1/2)d_{1,2,6}]\beta_2^2 \\ &+ (15d_{2,3,5} - (5/2)d_{1,3,6})\beta_2 + d_{2,3,6}, \\ \tilde{q}_1(\beta_2) := &(1/3)d_{1,2,6} + 5d_{1,3,5} + 5d_{1,2,5}\beta_2 - 25d_{1,2,3}\beta_2^3, \\ \tilde{q}_2(\beta_2) := &-25d_{1,2,3}\beta_2. \end{aligned}$$

Analogously we reduce equation (19) to an equation in β_2, β_3 by eliminating β_5 and then β_4. Using (15), we obtain

$$\begin{aligned} 0 = &(5\beta_3^2 + 15\beta_2^2\beta_3)d_{1,2,3} + (15\beta_2^2 + 5\beta_3)d_{5,1,2} \\ &+ 3\beta_2(d_{6,1,2} + \frac{5}{2}d_{5,1,3}) + \frac{d_{6,1,3}}{2} + \frac{15}{2}\beta_2\beta_4 d_{1,2,3}, \end{aligned}$$

and then by (14)

$$\begin{aligned} 0 = &d_{1,2,3}(-15\beta_2^4 + 30\beta_3\beta_2^2 + 10\beta_3^2) + d_{5,1,2}(15\beta_2^2 + 10\beta_3) \\ &+ \beta_2(15d_{5,1,3} + 5d_{6,1,2}) + d_{6,1,3}. \end{aligned} \tag{21}$$

In order to get rid of the term with β_2^4 in (21), we make a final substitution

$$\beta_3 := \bar{\beta}_3 + \alpha\beta_2^2, \qquad \alpha := (3/2) + \sqrt{15}/2.$$

Then

$$-15\beta_2^4 - 30\beta_3\beta_2^2 + 10\beta_3^2 = [-30\beta_2^2 + 20\alpha\beta_2^2]\bar{\beta}_3 + 10\bar{\beta}_3{}^2,$$

and (21) simplifies to

$$0 = p_0(\bar{\beta}_3) + p_1(\bar{\beta}_3)\beta_2 + p_2(\bar{\beta}_3)\beta_2^2, \tag{22}$$

where

$$
\begin{aligned}
p_0(\bar\beta_3) &:= 10d_{1,2,3}\bar\beta_3^{\,2} + 10d_{1,2,5}\bar\beta_3 + d_{1,3,6},\\
p_1(\bar\beta_3) &:= 5d_{1,2,6} + 15d_{1,3,5},\\
p_2(\bar\beta_3) &:= 10\sqrt{15}d_{1,2,3}\bar\beta_3 + (15+10\alpha)d_{1,2,5}.
\end{aligned} \tag{23}
$$

With the new variable $\bar\beta_3$, (20) transforms into

$$0 = q_0(\beta_2) + q_1(\beta_2)\bar\beta_3 + q_2(\beta_2)\bar\beta_3^{\,2}, \tag{24}$$

with

$$
\begin{aligned}
q_0(\beta_2) &:= \tilde q_0(\beta_2) - 25\alpha^2 d_{1,2,3}\beta_2^5 + \alpha\beta_2^2\tilde q_1(\beta_2),\\
q_1(\beta_2) &:= \tilde q_1(\beta_2) - 50d_{1,2,3}\alpha\beta_2^3,\\
q_2(\beta_2) &:= \tilde q_2(\beta_2) = -25d_{1,2,3}\beta_2,
\end{aligned}
$$

and the $\tilde q_i(\beta_2)$ defined as above. We summarize all this in

Lemma 2. *Under the assumption $d_{1,2,3} \neq 0$, i.e. assumption (11), and the normalization $\phi(0) = 0, \beta_1 := \phi'(0) = 1$, the vector $\beta_2, ..., \beta_7$ is a solution of the equations (22), (24) iff β_2, β_3 is a solution of the 2×2 system (22), (24).*

Remark: The systems (8)–(9) in the planar case, and (22)–(24) in the space case possess a similar structure. In (8) and (22) the coefficients of β_2 are polynomials of the same degree in β_3 and β_2, respectively. The same is true for (9) and (24), except that the corresponding polymials have different degrees.

§3. Existence Theorems

In view of the last remark, we treat in detail only the planar case.

Theorem 1. *The system (8)–(9) has at least one and at most 5 (real) solution pairs β_2, β_3 outside the line $\beta_3 = d_{4,1}/2d_{2,1}$.*

Proof: Let us write for shortness $x := \beta_2$ and $y := \beta_3$ as well as

$$A(y) := p_0(y), \quad 2B := p_1(y) = 60d_{4,2}, \quad C(y) := p_2(y) = 15d_{4,1} - 30d_{2,1}y.$$

Then (8) reads $0 = A(y) + 2Bx + C(y)x^2$. Formal solution for x gives

$$x = \varphi_\pm(y) := \frac{-B \pm \sqrt{B^2 - A(y)C(y)}}{C(y)} := \frac{-B \pm \sqrt{R(y)}}{C(y)}, \tag{25}$$

with the cubic polynomial

$$R(y) = 15[20d_{2,1}^2y^3 + 30d_{4,1}d_{2,1}y^2 + (2d_{6,1}d_{4,1} - 20d_{4,1}^2)y + 240d_{4,2}^2 - d_{4,1}d_{6,1}].$$

Then write (9) as $0 = \sum_{i=0}^3 a_i x^i + b_0 y + b_1 xy + b_2 xy^2$, and insert (25), since by assumption $C(y) \neq 0$. After multiplication with $C(y)^3$, we obtain

$$
\begin{aligned}
0 = &\sum_{i=0}^{3} a_i(-B \pm \sqrt{R(y)})^i C(y)^{3-i} + b_0 y C(y)^3\\
&+ b_1(-B \pm \sqrt{R(y)})C(y)^2 + b_2 y^2(-B \pm \sqrt{R(y)})C(y)^2.
\end{aligned} \tag{26}
$$

Now observe that

$$(-B \pm \sqrt{R(y)})^2 = B^2 + R(y) \pm 2B\sqrt{R(y)},$$
$$(-B \pm \sqrt{R(y)})^3 = -B^3 \pm 3B^2\sqrt{R(y)} - 3BR(y) \pm R(y)\sqrt{R(y)},$$

and sort all terms with and without $\sqrt{R(y)}$, respectively. Then (26) can be written as

$$U(y) = \pm V(y)\sqrt{R(y)}, \tag{27}$$

where $U(y)$ is a polynomial with leading term $-11 \cdot 15 \cdot \cdot 9000\, d_{4,2}\, d_{2,1}^3\, y^4$ and $V(y)$ also of degree 4 with leading term $9000\, d_{2,1}^3\, y^4$.

Hence under the above assumption, β_2, β_3 is a solution of the system (8)–(9) iff $y = \beta_3$ is a solution of (27) with sign either + or - on the right hand side. Suppose that n_1 and n_2 are the numbers of solutions of these two equations (including multiplicities). Then the squared equation

$$U^2(y) = V^2(y) \cdot R(y)$$

is of degree 11, and has $2(n_1 + n_2)$ solutions. Hence we conclude $n_1 + n_2 \leq 5$.

To prove existence, write (9) as

$$H(\beta_2, \beta_3) = 0, \tag{28}$$

where H is of degree 3 in β_2 and with $-10\beta_2\beta_3^2 d_{2,1}$ as leading term in β_3. Then introduce y^* as the largest zero of $R(y)$, so that in view of $R(+\infty) = +\infty$

$$R(y^*) = 0, \qquad R(y) > 0 \quad \text{for} y^* < y < \infty.$$

Now insert <u>both</u> functions $\varphi_\pm(y)$ in (28), and obtain the functions

$$H_{\pm}(y) := H(\varphi_{\pm}(y), y).$$

In order to guarantee existence of a solution of (8)–(9), it suffices therefore to show that the ranges of $H_{\pm}$ satisfy

$$H_+[y^*, \infty) \cup H_-[y^*, \infty) = \mathbb{R}. \tag{29}$$

For this, observe at first the properties

$$\varphi_{\pm}(y^*) = -B/C(y^*) = 2d_{4,2}/(d_{4,1} - 2y^*\ d_{2,1}) := \varphi^*$$

and

$$\varphi_{\pm}(y) \approx \pm|\frac{y}{3}|^{1/2}\text{sign }(d_{2,1}), \quad y \to \infty.$$

Then we distinguish the cases (assume without loss $d_{2,1} > 0$):

$$\text{i)} \quad y^* > \tilde{y} := d_{4,1}/2d_{2,1}, \qquad \text{ii)} \quad y^* < \tilde{y}.$$

In case i), we consider the ranges of $\varphi_\pm$ for (y^*, ∞), and have

$$\varphi_+[y^*, \infty) = (-\infty, \varphi^*], \qquad \varphi_-[y^*, \infty) = [\varphi^*, \infty). \tag{30}$$

Since both $\varphi_\pm$ are well defined and continuous on (y^*, ∞), so are the functions $H_\pm(y)$, and furthermore

$$H_\pm(y^*) = H(\varphi^*, y^*).$$

In combination with

$$H_\pm(y) \approx 10|d_{2,1}||y|^2\varphi_\pm(y), \quad y \to \infty, \tag{31}$$

it follows that

$$H_+[y^*, \infty) = (-\infty, H(\varphi^*, y^*)], \qquad H_-[y^*, \infty) = [H(\varphi^*, y^*), \infty),$$

and hence the desired assertion (29) in case i).

In case ii), the function $\varphi_-(y)$ has a singularity at $\tilde{y}$ which lies in (y^*, ∞). However, we can restrict its domain to $[y^*, \tilde{y})$ and still have $(d_{2,1} > 0)$

$$\varphi_-[y^*, \tilde{y}) = [\varphi^*, \infty). \tag{32}$$

On the other hand the function $\varphi_+(y)$ remains continuous on the whole interval (y^*, ∞) since

$$\varphi_+(y) = A(y)/(B + \sqrt{B^2 - A(y)C(y)}).$$

Hence we have

$$\varphi_+[y^*, \infty) = (-\infty, \varphi^*],$$

so that together with (32) we have the same situation as in (30) and can proceed further exactly as before in order to prove (29). □

It remains to discuss whether there exist solutions of (8)–(9) if $\beta_3 = d_{4,1}/2d_{2,1} := \tilde{y}$. In this case, (8) gives $\beta_2 = A(\tilde{y})/B$ if $B \neq 0$, and (9) can have a solution only under some additional constraint on the parameters $d_{2,1}, d_{4,1}, d_{4,2}, d_{6,1}, d_{6,2}$. We omit it here, as well as the one which results from (8) if in addition $B = 60d_{4,2} = 0$.

Further, we remark that there can indeed exist 5 solutions of (8)–(9). To this end, one can consider the case $d_{4,1} = 0$, where these equations simplify in such a way that solving (8) for $z := \beta_3 - (3/2)\beta_2^2$ gives

$$\pm z = \frac{3}{2}\beta_2^2 - \frac{2d_{4,2}\beta_2^{-1}}{d_{2,1}} - \frac{d_{6,1}\beta_2^{-2}}{30d_{2,1}} + O(\beta_2^{-4}).$$

Inserting this into (9) with sign +, it follows that

$$0 = -90d_{2,1}\beta_2^5 + 255d_{4,2}\beta_2^2 + 5d_{6,1}\beta_2 + d_{6,2} + O(1/\beta_2).$$

Here the polynomial of degree 5 dominates for large β_2, and it is clear that the 4 parameters $d_{2,1}, d_{4,2}, d_{6,2}, d_{6,1}$ can be chosen such that there exist 5 zeros outside some bounded interval containing 0. The situation for the system (22)–(24) in the space case is similar.

Theorem 2. *The system (22)–(24) has at least one and at most 7 (real) solution pairs $\beta_2, \bar{\beta}_3$ if $\bar{\beta}_3 \neq -(6+\sqrt{15})d_{1,2,5}/5\sqrt{15}d_{1,2,3}$.*

Proof: Concerning existence, the argument is the same as in Theorem 1. We define $x := \beta_2, y := \bar{\beta}_3$ and $A(y) := p_0(y), B := p_1(y), C(y) := p_2(y)$. Then $R(y)$ in (25) has the form

$$R(y) = 50\sqrt{15}d_{1,2,3}^2 y^3 + 50\sqrt{15}d_{1,2,5}y^2 \ + \text{ linear term.}$$

Again define $\tilde{y} := -(6+\sqrt{15})d_{1,2,5}/5\sqrt{15}d_{1,2,3}$ as the zero of $C(y)$ and y^* as the largest zero of $R(y)$, and let $\varphi_\pm(y) := (B \pm \sqrt{R(y)})/C(y)$. Its asymptotic behaviour is described by

$$\varphi_\pm(y) \approx \text{sgn } d_{1,2,3}\left|\frac{2y}{\sqrt{15}}\right|^{1/2}, \quad y \to \infty. \tag{33}$$

Now we distinguish as in Theorem 1 the cases i) and ii) and conclude that either (30) holds or (31), respectively.

Next write (24) similarly as in (28) as

$$\tilde{H}(\beta_2, \bar{\beta}_3) = 0$$

and define $\tilde{H}_\pm(y) := \tilde{H}(\varphi_\pm(y), y)$. Its asymptotic behaviour is somewhat more complicated to determine than in (31), since by the definition of the $q_i(y)$, $i = 0, 1, 2$, in (24), the leading term of $\tilde{H}(\beta_2, \bar{\beta}_3)$ now has the form

$$(\frac{15}{2} - 25\alpha^2)d_{1,2,3}\beta_2^5 + 25\alpha d_{1,2,3}\beta_2^5 + (25d_{1,2,3} - 50\alpha d_{1,2,3})\bar{\beta}_3\beta_2^3 - 25d_{1,2,3}\beta_2\bar{\beta}_3{}^2.$$

Inserting (33) with $\bar{\beta}_3 = y$, one derives from this

$$\tilde{H}_\pm(y) \approx \text{const. } |y|^{5/2}, \quad y \to \infty.$$

Hence we obtain the same property (29) for $\tilde{H}_\pm$ as for $H_\pm$ in Theorem 1, and existence of a (real) solution pair of (22), (24) follows.

In order to show the bound on the number of solutions, we proceed again as in Theorem 1. We solve (22) for $x = \beta_2$ as in (25) and insert it into (24) written as

$$0 = \sum_{i=0}^{5} a_i x^i + y \sum_{i=0}^{3} b_i x^i + cxy^2,$$

with constants a_i, b_i and c. This gives after multiplication with $C(y)^5$

$$\begin{aligned} 0 &= \sum_{i=0}^{5} a_i(-B \pm \sqrt{R(y)})^i C(y)^{5-i} \\ &+ y\sum_{i=0}^{3} b_i(-B \pm \sqrt{R(y)})^i C(y)^{5-i} + cy^2(-B \pm \sqrt{R(y)})C(y)^4. \end{aligned}$$

Then sort by terms with and without $\pm\sqrt{R(y)}$. The result is an equation of type (27), this time with polynomials $U(y)$ of degree 7 and $V(y)$ of degree 6. Therefore we conclude by the same argument as in Theorem 1 that the system (22), (24) has at most 7 solutions . □

The discussion of the degenerate case $\bar{\beta}_3 \neq -(6+\sqrt{15})d_{1,2,5}/5\sqrt{15}d_{1,2,3}$ is omitted. It can be done similarly as for Theorem 1.

§4. Final remarks

We have shown that local approximation order 8 can be achieved with parametric polynomial curves of degree 4 in the planar case, and degree 5 in the space case. The method for this consists in determining a suitable reparametrization ϕ, and then the Taylor-polynomial with respect to $\underline{f} \circ \phi$. For practical purposes, however, it is important to consider also Lagrange or Hermite interpolation in the sense of the equations (1). This has been done in case $k = 2$ for $d = 2$ in [1,3,6] and for $d = 3$, in [5], but results for higher k do not seem to be available so far. In this respect another interesting open question is which order of geometric continuity can be preserved when a piecewise polynomial curve is constructed by pieces of such local aproximations.

References

1. de Boor, C., K. Höllig, and M. Sabin, High accuracy geometric Hermite interpolation, Comput. Aided Geom. Design **4** (1987), 269–278.
2. Degen, W. L. F., High accuracy approximation of parametric curves, in *Mathematical Methods for Curves and Surfaces*, M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville, 1995, 83–98.
3. Feng, Y. Y. and J. Kozak, On G^2 continuous cubic spline interpolation, BIT **37** (1997), 312–332.
4. Gregory, J. A., Geometric Continuity, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker (eds.), Academic Press, New York, 1989, 353–371.
5. Höllig, K. and J. Koch, Geometric Hermite interpolation, Comput. Aided Geom. Design **12** (1995), 567–580.
6. Mørken, K. and K. Scherer, A general framework for high-accuracy parametric interpolation, Mathematics of Computation **66** (1997), 237–260.
7. Rababah, A., High order approximation method for curves, Comput. Aided Geom. Design **12** (1995), 89–102.
8. Schaback, R., Geometrical differentiation and high-accuracy curve interpolation, in *Approximation Theory, Spline Functions and Applications*, S. P. Singh (eds.), Kluwer Academic Publishers 1992, 581–584.
9. Scherer, K., On local parametric approximation by polynomial curves, in *Approximation Theory*, M. W. Mueller, M. Felten, and D. H. Mache (eds.), Akademie-Verlag, 1995, 285–292.

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